The echo effect in a Josephson junction array

1995 J. Phys.: Condens. Matter 7 9833

(http://iopscience.iop.org/0953-8984/7/50/017)

J. Phys.: Condens. Matter 7 (1995) 9833–9844. Printed in the UK

The echo effect in a Josephson junction array

V V Kurin and A V Yulin

Institute for the Physics of Microstructure of the Russian Academy of Science, 46 Ulyanov Street, 603600 Nizhny Novgorod, Russia

Received 16 June 1995, in final form 23 October 1995

Abstract. The echo effect in self-excited oscillator arrays and in a Josephson junction array has been theoretically investigated. The required parameters of the Josephson array and the limiting time due to the influence of thermal noise have been estimated.

1. Introduction

It is well known that an echo effect can be found in different physical systems. This effect manifests itself through a delayed response of the system influenced by two external pulses. For instance, for a spin system two pulses of external radiation at a frequency close to the resonance one give rise to a pulse of spontaneous spin radiation. Since 1950 when this effect was discovered in a spin system by Hahn [1], a great variety of echo effects have become known—for example, the echo in plasma [2], the cyclotron echo [3] and the photon echo [4]. Recently this effect has been discovered in quantum systems [5]. It is inherent in such physical systems where an observable macroscopic response arises from a sum of many independent contributions from different 'particles' (molecules, oscillators, spins, etc) and where damping of the macroscopic response, also referred to as collisionless damping, occurs through phase scattering [7, 6], but not through true thermodynamic damping. Since individual 'particles' store the information about their initial phases, by affecting this system via some external action, it is possible to return the system to a state close to the initial one, when the macroscopic response existed, and to have an echo response.

In this paper we would like to focus on another set of systems which can be classified as the systems with phase memory—and, therefore, as will be shown, could demonstrate an echo effect. The systems which we intend to discuss are arrays of Josephson junctions which have been widely investigated theoretically [8] and experimentally [9, 10, 11]. From the general dynamics point of view a Josephson junction array is a set of self-excited oscillators and, therefore, the echo effect may be found in sets of self-excited oscillators of any physical origin. In the absence of thermal noise self-excited oscillators have an infinite phase memory time and, therefore, the echo effect can probably be observed easily.

This paper is devoted to the investigation of the echo effect in Josephson junction arrays and in arrays of self-excited oscillators.

We intend to demonstrate that an echo effect can occur in any system consisting of noninteracting self-excited oscillators, if changes in the phases of these oscillators under the action of an external pulse depend on the oscillator phases. In order to demonstrate this, the echo effect has been considered in general terms in section 2 for the simplest system of noninteracting self-excited oscillators without noise. In section 3 using the results obtained

0953-8984/95/509833+12$19.50

we investigate the echo effect in a Josephson array in a resistive state. The problem of Josephson junction dynamics is too complicated to solve analytically, if we take into account thermal noise and mutual interaction between the junctions; but for the realistic parameters discussed in section 3 we can neglect the mutual interaction. The influence of a weak thermal noise during the external pulse can be also neglected because this pulse is supposed to be short. The influence of the noise at other times leads only to a loss of phase memory, so the echo effect can be observed only at finite times. To avoid technical difficulties we first consider the pure dynamical problem and demonstrate the echo effect in the Josephson junction array without noise. In the next section we estimate the influence of a thermal noise and find out the times for which the echo effect can be observed. In the last section we briefly summarize the main results obtained in the paper.

2. General considerations

To demonstrate the existence of an echo effect in Josephson arrays and arrays of self-excited oscillators we assume that we have a set of independent self-excited oscillators making quasiharmonic oscillations at close frequencies. To describe the collective dynamics of these oscillators let us introduce a distribution function of oscillators of frequency ω and phase ψ, $f(\psi, \omega, t)$. This distribution function is normalized to satisfy the following relation:

$$\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} f(\omega, \psi)\,\mathrm{d}\omega\,\mathrm{d}\psi = 1. \tag{1}$$

Sometimes it is useful to define a new variable Δ, the deviation of frequency $\Delta = \omega - \bar{\omega}$, where $\bar{\omega}$ is the average frequency of the array:

$$\bar{\omega} = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} \omega f(\omega, \psi)\,\mathrm{d}\psi\,\mathrm{d}\omega \tag{2}$$

and the array will be characterized by the distribution function $f(\Delta, \psi, t)$. Using this function we can find any averaged physical quantity—for example, we can find the array radiation intensity $P(t)$, which will demonstrate the echo behaviour:

$$P(t) = N^2 \left| \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} f(\Delta, \psi, t)\mathrm{e}^{\mathrm{i}\psi}\,\mathrm{d}\psi\,\mathrm{d}\Delta \right|^2. \tag{3}$$

Here N is the number of oscillators in the array. The current phase ψ in the absence of any external influence is determined by the relation $\psi = \psi_0 + \Delta t$ which is the characteristic equation for the following kinetic equation, which the distribution function $f(\Delta, \psi)$ in the absence of noise obeys:

$$\frac{\partial f}{\partial t} + \Delta\frac{\partial f}{\partial \psi} = 0. \tag{4}$$

The solution of this equation has the form

$$f(\Delta, \psi, t) = f_0(\Delta, \psi - \Delta t) \tag{5}$$

where $f_0 = f(t = 0)$ represents the initial condition for equation (4). Using these relations we could rewrite the expression (3) for radiated power of the array of free (unaffected by external radiation and not mutually interacting) self-excited oscillators as

$$P(t) = N^2 \left| \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} f(\Delta, \psi_0)\mathrm{e}^{\mathrm{i}(\psi_0+\Delta t)}\,\mathrm{d}\psi_0\,\mathrm{d}\Delta \right|^2. \tag{6}$$

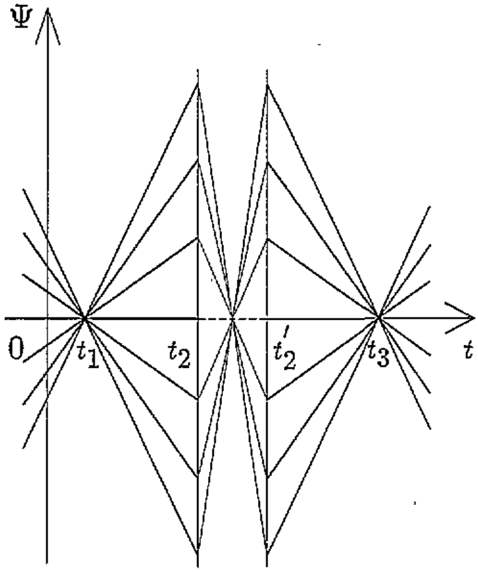

Figure 1. The dynamics of the relative phase difference is shown. At the time $t = t_1$ all oscillators are in phase. After that, until $t = t_2$, phase scattering takes place. During the time range $t_2 < t < t_2'$ a second pulse affects the system; $t_2' = t_2 + \tau_2$ where τ_2 is duration of the second pulse. At the time $t_3 = 2t_2 + \tau_2 - t_1$ the system has its coherence restored.

Now we consider qualitatively the problem of an echo arising in an array of independent self-excited oscillators. To demonstrate the reversibility in this system we suppose that oscillators in the array are free all the time except for in two short intervals $(t_1 - \tau_1, t_1)$ and $(t_2, t_2+\tau_2)$, during which two pulses of external radiation with durations τ_1, τ_2, respectively, affect the array. Schematically this situation is shown in figure 1. In this figure the first pulse makes all the oscillators in phase. After that phase scattering takes place and the macroscopic response reduces until the time $t = t_2$ is reduced. The second pulse changes the relative phases Ψ of the oscillators so that for $(n-1)\pi < \Psi(\text{just before the pulse}) < (n+1)\pi$ we have

$$\Psi(\text{straight after the pulse}) = 2n\pi - \Psi(\text{just before the pulse})$$

where n is an integer. Due to this phase correction, at the time $t_3 = 2t_2 + \tau_2 - t_1$ all oscillators will be in phase again, which gives rise to an echo pulse. The general case is considered analytically below.

Suppose that at the initial moment the system is in a state with the following distribution function:

$$f(\Delta, \psi_0) = \frac{1}{2\pi} f(\Delta) \tag{7}$$

where $f(\Delta)$ is the frequency distribution function of oscillators. We can take this, for example, in the form

$$f(\Delta) = \frac{1}{(2\pi)^{1/2}\Delta_0} \exp - \frac{\Delta^2}{2\Delta_0^2}.$$

For this kind of distribution the macroscopic response is zero due to a uniform distribution of phase. It is easy to see that this distribution will remain unchanged up to t_1, since it satisfies the kinetic equation (4). The effect of external pulse action on each oscillator in the array can be described in terms of the mapping

$$\psi(\text{just before the pulse}) \Rightarrow \psi(\text{straight after the pulse})$$

which, in each particular case, should be defined from the dynamical equations describing the behaviour of the self-excited oscillator under external influence. In this section we will assume this mapping to be a known function and denote it as $\psi(t_1+\tau_1)=\Phi_1(\psi(t_1))$ for the first pulse and $\psi(t_2+\tau_2)=\Phi_2(\psi(t_2))$ for the second one. Note that these functions are periodic with a period 2π. Now, using the functions introduced, we can write the dependence of the current phase on time for each oscillator in the form

$$\psi(t)=\begin{cases}\psi_0+\Delta t & \text{for } t<t_1\\ \Phi_1(\psi(t_1))+\Delta(t-t_1) & \text{for } t_1<t<t_2\\ \Phi_2(\psi(t_2))+\Delta(t-t_2) & \text{for } t>t_2.\end{cases}$$

Here, assuming the duration of pulses to be small enough that the conditions $\Delta\tau_{1,2}\ll\pi$ are satisfied for every oscillator, we consider the times $t_{1,2}$ and $t_{1,2}+\tau_{1,2}$ to be equal.

Since the phase distribution function of oscillators (7) does not depend on time until the time t_1, and hence we have $f(\Delta,\psi_0)=f(\Delta,\psi_1)$, where $\psi_1=\psi(t_1)$, we can assume that $t_1=0$, or, in other words, average over ψ_1 instead of ψ_0. Thus, in order to obtain the intensity of radiation after the second pulse, we should calculate the integral that follows from (6):

$$P(t)=\left(\frac{N}{2\pi}\right)^2\left|\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}f(\Delta)\mathrm{e}^{\mathrm{i}(\Phi_2(\Phi_1(\psi_1)+\Delta T)+\Delta t)}\,\mathrm{d}\psi_1\,\mathrm{d}\Delta\right|^2 \tag{8}$$

where we have introduced a new variable $T=t_2-t_1$ and changed the time origin via $t=t-t_2$. Since Φ_2 is a periodic function of its argument (and so is Φ_1), we can expand $\mathrm{e}^{\mathrm{i}\Phi_2(\Phi_1(\psi_1)+\Delta T)}$ into a Fourier series:

$$\mathrm{e}^{\mathrm{i}\Phi_2(\Phi_1(\psi_1)+\Delta T)}=\sum_{l=-\infty}^{\infty}C_l\mathrm{e}^{\mathrm{i}l(\Phi_1(\psi_1)+\Delta T)}. \tag{9}$$

Using this expansion we can represent the array radiation intensity as

$$P=\left(\frac{N}{2\pi}\right)^2\sum_{l=-\infty}^{\infty}\left|C_l\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}l\Phi_1(\psi_1)}\,\mathrm{d}\psi_1\right|^2\left|\int_{-\infty}^{\infty}f(\Delta)\mathrm{e}^{\mathrm{i}((l-1)T+t)\Delta}\,\mathrm{d}\Delta\right|^2 \tag{10}$$

which shows that the response of an array of self-excited oscillators to two pulses is a series of pulses of similar shape, occurring at times $t\approx(1-l)T$, for $l\leqslant-1$. The pulses have magnitudes

$$D_l=\left|C_l\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}l\Phi_1(\psi)}\,\mathrm{d}\psi\right|^2$$

and their shape is determined by the second integral in (10).

In the next section, using the results obtained above we consider a particular case of the echo effect in a Josephson junction array affected by external microwave radiation.

3. The echo effect in a Josephson array

In this section we apply the general theory developed above to investigation of an echo effect in Josephson junction arrays. Here we do not take into account the noise in the system, and we solve a purely dynamical problem.

Because all the distinctive features of this case lie in the functions Φ introduced in the previous section, which describe the mapping of the phase ψ before the external pulse into the phase $\bar{\psi}$ straight after the pulse, $\bar{\psi} = \Phi(\psi)$, all we need to do is to determine the particular form of this mapping function.

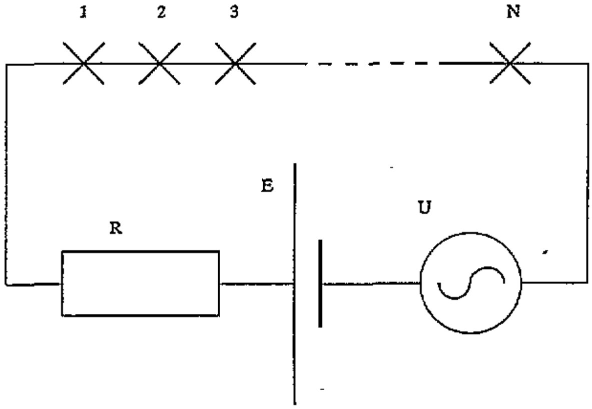

Figure 2. The Josephson junction (1–N) series array driven by DC and AC voltage sources through the load R.

Now let us consider a simple example of a one-dimensional Josephson array—a set of series-connected junctions with an external load (see figure 2)—and suppose for simplicity that all junctions have equal critical current I_c and capacitance C but different normal resistances r_i. Within the framework of the resistive shunted junction (RSJ) model such an array can be described by equations which in dimensionless variables have the following form:

$$\beta\ddot{\varphi}_i + \gamma_i\dot{\varphi}_i + \sin\varphi_i = I_0 + s(t) - \alpha\sum_{j=1}^{N}\dot{\varphi}_j \qquad (11)$$

where φ_i is the Josephson phase difference of the ith junction, $I_0 = E/RI_c$ and $s(t) = U/RI_c$ are the dimensionless bias direct current and the external pulse current, respectively, parameters $\gamma_i = \bar{r}/r_i$ represent normal resistance of the ith junction, and $\bar{r}$ is the averaged resistance defined by relation $\bar{r}^{-1} = \langle 1/r\rangle$, so $\bar{\gamma} = 1$. The coefficient $\alpha = \bar{r}/R$ determines the strength of the interaction between junctions through the common load, R is the resistance of the load, $\beta = 2\pi\bar{r}^2 I_c C/\Phi_0$ is the McCumber parameter ($\Phi_0 = h/2e$ is the quantum of magnetic flux), and the time current and voltage are normalized to an average 'gap' frequency $\Omega_g = 2\pi\bar{r}I_c/\Phi_0$, with critical currents I_c and $\bar{r}I_c$, respectively.

When the current in an array, I_0, exceeds the critical current of the junctions, all junctions in the array get into a resistive state and oscillate with a frequency $\omega_\iota = \omega_i(I) \approx I_0/\gamma_i$. In this scheme the power dissipated in the load resistor or the power of radiation, if the load is a matched antenna, is equal to

$$P = \frac{1}{2}I_\sim^2 R \approx \frac{\bar{\gamma}^2}{2R}\sum_{i=1}^{N}\dot{\varphi}_\sim^2 \qquad (12)$$

where $I_\sim$ is the amplitude of the alternating current through the array, and $\dot{\varphi}_\sim$ is the amplitude of the alternating voltage on an individual junction.

Let us introduce the array parameters at fixed I_0—the average frequency $\bar{\omega}$ and dissipation constant $\bar{\gamma}$, and spread of frequency Δ_0:

$$\bar{\gamma} = \sum_{i=1}^{N} \gamma_i/N \qquad \bar{\omega} = \sum_{i=1}^{N} \omega_i/N \qquad \Delta_0^2 = \sum_{i=1}^{N} (\omega_i - \bar{\omega})^2/N. \tag{13}$$

To demonstrate the possibility of an echo effect occurring in Josephson arrays and to obtain an analytical solution of the problem, we assume that the following conditions are satisfied:

$$\frac{\alpha N}{\sqrt{(\beta\bar{\omega}^2 + 1)}} \ll \Delta_0 \qquad \Delta_0/\bar{\omega} \ll 1 \qquad I_0 \gg 1. \tag{14}$$

This assumption allows us to neglect the mutual interaction between junctions in the array and apply a perturbation method. It should be noted here that, if the first of conditions (14) is satisfied, then corrections of oscillator phases are much smaller than unity at all times and, hence, we can completely neglect the mutual interaction between the junctions.

Now, following the prescriptions of the general theory, we introduce the phase and frequency distribution function $f(\Delta, \psi)$ of junctions in the array, where $\Delta = \omega - \bar{\omega}$ is the frequency deviation from the averaged value $\bar{\omega}$, and ψ is the slow phase of oscillation introduced by the relation

$$\varphi = \bar{\omega}t + \psi + \operatorname{Im} b\, e^{i\bar{\omega}t}. \tag{15}$$

Here φ is the Josephson phase difference of the individual junction which obeys equation (11), and b is the complex amplitude of the phase oscillations. Now, in order to obtain explicit relations for the echo response, assume that external pulses have the form

$$s(t) = \operatorname{Im} a(t)\, e^{i\bar{\omega}t} = A(t)\sin(\bar{\omega}t + \chi(t)) \tag{16}$$

where $a = Ae^{i\chi}$ is the complex amplitude of external radiation, which is assumed smooth enough on the scale of ω^{-1} that $|\dot{a}| \ll \omega|a|$. If, besides, the spread of frequency in the array is rather small, $\Delta_0 \ll \bar{\omega}$, the phase ψ obeys the following reduced equation [12]:

$$\beta\ddot{\psi} + \gamma\dot{\psi} + \Omega^2(t)\sin(\psi + \chi_1 + \chi) = \gamma\Delta. \tag{17}$$

Here we denote $J_1(B)$ (J_1 is the first-order Bessel function) as $\Omega^2(t)$, where $b = Be^{i\chi_1}$ is the complex amplitude of an alternating phase defined above (15) which is expressed as

$$b = \frac{a}{-\beta\bar{\omega}^2 + i\gamma\bar{\omega}}.$$

We will need the explicit forms for B and χ_1:

$$B = \frac{A}{\bar{\omega}\sqrt{\beta^2\bar{\omega}^2 + \gamma^2}} \qquad e^{i\chi_1} = -\frac{-\beta\bar{\omega} - i\gamma}{\sqrt{\beta^2\bar{\omega}^2 + \gamma^2}}. \tag{18}$$

Note that the coefficient γ relates to the oscillation frequency via $\Delta = I(1/\gamma - 1)$. We recall that in our units $\bar{\gamma} = 1$. If the external pulse duration is large enough, then, as can be seen from (17), synchronization of Josephson junctions by external radiation can take place. This leads to formation of Shapiro steps in the current–voltage characteristic of the junction [13]. But a short pulse only changes the phase of the junction and this phase correction can cause the echo effect.

After the pulse, when $\Omega(t) = 0$, equation (17) takes the form

$$\beta\ddot{\psi} + \gamma\dot{\psi} = \gamma\Delta \tag{19}$$

whose solution quickly—in a time of order β (the distance between pulses t_1 is assumed to be much larger than β)—tends to the solution of the reduced equation $\dot{\psi} = \Delta$ which is the characteristic equation for the free kinetic equation (4) introduced in section 2. Therefore, in order to get the mapping produced by the external pulse we ought to take into account not only the dynamics within the pulse, described by (17), but also the relaxation process straight after the pulse, described by (19).

Equation (17) is too complicated to solve analytically, so to obtain the mapping function a computer simulation and qualitative investigation is required. It can be solved easily in two limiting cases. The first one is the case of rather strong and long external pulses such that $\Omega^2 > \gamma\Delta_0$ and $\tau^{-1} \ll (\gamma/2)(1-\sqrt{1-4\beta^2\Omega^2/\gamma^2})$, which make all junctions synchronous with the external radiation. The second limiting case is the case of weak external influence, where we can determine a pulse-induced change of phase using the perturbation theory.

Now consider the first case of synchronizing the pulse and obtaining the mapping. If $\Omega^2 \gg \gamma\Delta_0$, we can expand the 'sines' in equation (17) into a power series and, by taking into account only the linear term, derive the linear equation

$$\beta\ddot{\psi} + \gamma\dot{\psi} + \Omega^2(\psi + \chi_1 + \chi) = \gamma\Delta \tag{20}$$

which is easy to solve. It should be remembered that we are solving the equation for an individual junction. Its solution is

$$\chi_1 + \chi + \psi - \gamma\Delta/\Omega^2 = \mathrm{Im}\{c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t}\} \qquad \dot{\psi} = \mathrm{Im}\{c_1\gamma_1 e^{\gamma_1 t} + c_2\gamma_2 e^{\gamma_2 t}\} \tag{21}$$

where $\gamma_{1,2} = -(\gamma/2)(1 \pm \sqrt{1-4\beta^2\Omega^2/\gamma^2})$. Since just before the pulse $\dot{\psi} = \Delta$, relations (21) will determine ψ, $\dot{\psi}$ straight after the pulse, if we put $t = \tau$ in them. In other words, these relations represent the mapping in the parametric form. Besides this, we need to determine a phase change after the pulse within the relaxation time $\tau_{rel} \simeq \beta/\gamma$. The phase dynamics during this time obeys the equation

$$\beta\ddot{\psi} + \gamma\dot{\psi} = \gamma\Delta$$

from which we get

$$\psi(t_1 + \tau + t) = \Delta t + \frac{\dot{\psi}(t_1+\tau) - \Delta}{\gamma}(1 - e^{-(\gamma/\beta)t}) + \psi(t_1 + \tau). \tag{22}$$

If $(\gamma/\beta)t \gg 1$, we obtain

$$\psi(t_1 + \tau + t) = \Delta t + \frac{\dot{\psi} - \Delta}{\gamma} \tag{23}$$

which shows that the dynamics during the relaxation time results only in the addition of the term $(\dot{\psi} - \Delta)/\gamma$ to the mapping determined by the dynamics through the pulse duration. Thus, equations (21) and (3) yield jointly the full mapping

$$\psi(t_1 + \tau) = \Phi(\psi(t_1))$$

needed for the calculations. If the pulse is long enough, such that $\min\{\gamma_{1,2}\} \gg 1$, all phases become synchronous, which corresponds to the following mapping:

$$\psi(\text{after pulse}) = -(\chi_1(\Delta) + \chi). \tag{24}$$

The other limiting case allowing us to derive a simple expression for the mapping function is the case of a weak and short pulse, such that

$$\Omega^2 \ll \gamma\Delta \qquad \text{and} \qquad \frac{\Omega^2\tau}{\beta\gamma} \ll \pi. \tag{25}$$

If these conditions are satisfied, we can integrate equation (17) over the unperturbed trajectory $\psi = \Delta t$, which yields

$$\dot{\psi}(t+\tau) = \Delta - \frac{\Omega^2\tau}{\beta}\sin(\psi(t) + \chi_1 + \chi) \qquad \psi(t+\tau) = \psi(t) + \Delta\tau. \tag{26}$$

Further, using expression (3) for a phase change during the relaxation time, we find finally the expression for the mapping produced by the weak pulse:

$$\psi(t_1 + \tau + \tau_{rel}) = \psi(t_1) + \Delta(\tau + \tau_{rel}) - \frac{\Omega^2\tau}{\beta}\sin(\psi(t) + \chi_1 + \chi). \tag{27}$$

The second term simply reflects the phase accumulating due to the unperturbed dynamics, and the last one is the perturbation contributed by the external pulse.

Now, since we know the explicit form of the mapping, we can calculate the echo response using the expression derived in section 2. Suppose that our system is affected by two pulses. Assume further that the first one makes all junctions synchronous; then from (24) we obtain the distribution function in the phase:

$$f(\psi) = \int f[(\Delta(\chi_1)]\,\frac{\mathrm{d}\Delta}{\mathrm{d}\chi_1}\,\delta(\psi - \chi - \chi_1)\,\mathrm{d}\chi_1 \tag{28}$$

where $f(\Delta)$ is the frequency distribution function of the junctions. If $\bar{\omega} \gg \Delta_0$ then this function can be expressed in the explicit form

$$f_0(\psi) = \frac{1}{\sqrt{2\pi}\,\Delta_0}\exp\left\{-\frac{(\psi - \bar{\chi}_1)^2}{2\Delta^2\,(\partial\chi_1/\partial\Delta)^2}\right\}(\partial\chi_1/\partial\Delta)^{-1}. \tag{29}$$

Here we have taken the frequency distribution function in the form (7) and used a linear approximation for the dependency of the synchronization phase on the oscillator frequency, $\chi_1(\Delta)$, equation (18): $\chi_1 = \bar{\chi}_1 + (\partial\chi_1/\partial\Delta)\Delta$. In the high-frequency limit the derivative $\partial\chi_1/\partial\Delta$ is defined by the following expression:

$$\frac{\partial\chi_1}{\partial\Delta} = \frac{1}{I}\frac{\beta^2\omega^2}{\beta^2\omega^2 + \gamma^2}.$$

Note that if the synchronization phase χ_1 does not depend on Δ, for example in the case where $\gamma \ll \bar{\omega}$ when $\chi_1 = 0$, the phase distribution function transforms into

$$f(\psi) = \delta(\psi - \chi).$$

Everywhere above we considered the spread of the damping parameter γ to be appropriately small: $\gamma - \bar{\gamma} \ll \bar{\gamma}$, and using this assumption changed the averaged functions to functions of averaged variables.

Thus, according to relation (5), after the first synchronizing pulse, the frequency and phase distribution function has the following form:

$$f(\Delta, \psi, t) = f_0(\Delta, \psi - \Delta t) \tag{30}$$

where f_0 is given by (29) and we shifted the time origin to the end of the first pulse. Averaging $e^{i\psi}$ with respect to the distribution function (29) yields the time dependency of radiation power after the first pulse:

$$P(t) = N^2\left|\int_{-\infty}^{\infty} \mathrm{d}\psi\,\mathrm{d}\Delta\,f_0(\Delta, \psi - \Delta t)e^{i\psi}\right|^2 = N^2\left|\int_{-\infty}^{\infty} \mathrm{d}\psi\,\mathrm{d}\Delta\,f_0(\Delta, \psi_0)e^{i\psi_0 + \Delta t}\right|^2$$

$$= N^2\exp\left(-2\Delta^2\left[\left(\frac{\partial\chi_1}{\partial\Delta}\right)^2 + t^2\right]\right). \tag{31}$$

The first term in the exponent reflects a power decrease caused by the initial phase mismatch due to the dependency of the synchronization phase on the individual junction parameters; the second one is related to 'phase scattering' due to the spread in frequency of the self-excited oscillators.

Now suppose that at the time $t = T$ our system is affected by the second pulse, which is rather short and weak, such that the conditions (25) are satisfied and the mapping function has the form (27). Using the general representation of radiated power (8) we obtain the following expression:

$$P(t) = N^2 \left| \iint_{-\infty}^{\infty} f_0(\Delta, \psi_0) e^{i\Phi(\Delta, \psi_0 + \Delta T) + \Delta t} \, d\Delta \, d\psi_0 \right|^2 . \tag{32}$$

Further, expanding $e^{i\Phi(\psi)}$ into a Fourier series:

$$e^{i\psi - D\sin(\psi + \chi + \chi_1)} = e^{i\psi} \sum_{l=-\infty}^{\infty} J_l(D) e^{-il(\psi + \chi + \chi_1)}$$

(here the J_l are the Bessel functions, and we denote the parameter of the mapping by $D = \Omega^2\gamma/\tau$) and taking the integral we obtain the final expression for the echo response to the second pulse:

$$\begin{aligned} P(t) = N^2 &\left| \sum_{l=-\infty}^{\infty} J_l \right|^2 \\ &\times \left| \iint d\Delta \, d\psi_0 \, \exp\left(i[\psi_0 + \Delta(T+t)] - il(\psi_0 + \Delta T + \Delta \, \partial\chi_1/\partial\Delta)\right) \right|^2 \\ = N^2 &\left| \sum_{l=1}^{\infty} J_l \right|^2 \exp\left(-2\Delta_0^2[T(1-l) + t - \partial\chi_1/\partial\Delta]^2 \right. \\ &\left. -2\Delta_0^2(\partial\chi_1/\partial\Delta)^2(1-l)^2\right). \end{aligned} \tag{33}$$

Here, for calculations, we used the explicit form (29) for the distribution function and shifted the time origin to the end of the second pulse. We can see that a response of the Josephson array to the second pulse is essentially a sequence of echo pulses, taking place at the times $t = \partial\chi_1/\partial\Delta + T(l-1)$, their shapes being determined by the frequency distribution function of the junction. The value

$$N^2 \left| \sum_{l=-\infty}^{\infty} J_l(D) \right|^2 e^{-2\Delta_0^2(\partial\chi_1/\partial\Delta)^2(1-l)^2}$$

is the amplitude of lth echo pulse. A qualitative picture demonstrating the sequence of echo pulses is given in figure 3.

Now, using the formulas obtained above we can make some estimates of the amplitude and duration of echo pulses and discuss the possibility of experimental observation of the echo. For a typical 1D Josephson junction array [11] consisting of $N \approx 500$ junctions with $r \approx 0.4\ \Omega$ and $I_c \approx 3$ mA, connected to a matched load for the maximum output power we will have the expression

$$P_{max} \approx 0.1 N r (I_c)^2 \approx 0.2 \text{ mW}.$$

For the pulse duration we have the estimate $\tau \approx \Delta_0^{-1}$, which for the typical spread of oscillation frequency in the array (of order 1%) and for the averaged frequency

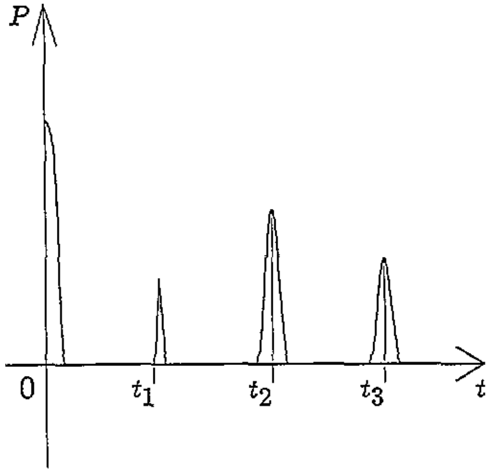

Figure 3. The power of radiation of the Josephson junction array as a function of time. At the time $t = 0$ all junctions are synchronized by the first long pulse. After this time phase scattering takes place and the power of the radiation decreases. At the time $t = t_1$, the second pulse affects the system and, because of this, at times $t_2 = T + \partial\chi_1/\partial\Delta, t_2 = 2T + \partial\chi_1/\partial\Delta$ (and so on) echo pulses arise.

($\bar{\omega}/2\pi \approx 100$ GHz) is about $\tau \approx 1$ ns. Such pulses are quite easy to observe unless thermal fluctuations destroy the phase coherency. So now we should estimate the role of thermal noise.

4. The influence of thermal noise

To consider the influence of thermal fluctuations on the echo effect let us suppose that our system is affected by an external fluctuation current I_f with a zero mean value $\langle I_f \rangle = 0$ and a correlation function

$$\langle I_f(t) I_f(t+\tau) \rangle = \frac{4\pi kT}{\Phi_0 I_c} \delta(\tau).$$

Here we use the dimensionless variables $I_f = I_f/I_c$, k is the Boltzmann constant, T is the temperature, I_c is the critical current, $r = \langle r_i^{-1} \rangle^{-1}$ is the averaged normal resistance, and τ is the dimensionless time normalized, as in section 1, to $\Omega = 2\pi r I_c/\Phi_0$, where Φ_0 is the flux quantum. The equation for a stochastic Josephson phase of an individual junction with averaged parameters has the form

$$\beta\ddot{\varphi} + \dot{\varphi} + \sin\varphi = I + I_f \tag{34}$$

whence, using the standard method (see, for example, [14]), we will have an estimate for the mean square value of the phase fluctuation characterizing a phase scattering rate. We give it in dimension variables:

$$\bar{\varphi} = \frac{I}{I_c}\Omega t \qquad \bar{\varphi^2} - \bar{\varphi}^2 = \frac{kT\Omega}{r I_c^2}\Omega t = \frac{4\pi^2 krT}{\Phi_0^2} t. \tag{35}$$

The echo effect can be observed only if the time interval between the first and the echo pulses does not exceed the phase scattering time t_{lim} when $|\varphi - \bar{\varphi}| \approx 2\pi$, so the limiting time of the echo effect t_{lim} is defined by the following expression:

$$t_{lim} \sim \frac{\Phi_0^2}{krT}.$$

Using the typical values of parameters of a low-temperature Josephson junction: $r \sim 0.1$–$1\ \Omega$, $T \sim 4$ K, we can estimate for τ_{lim}

$$\tau_{lim} \simeq 2\text{–}20 \text{ ns}$$

which is a rather limiting condition for observing an echo effect in Josephson arrays experimentally. Because both the intensity of the thermal noise and the normal conductivity of the Josephson junction depend on temperature, there is an optimal temperature at which the echo effect can be observed at the longest. To find the optimal temperature one has to find the global minimum of the function $r(T)T$ ($r(T)$ is the dependence of the junction normal conductivity on the temperature T). It is seen that the optimal temperature must not be zero. Therefore, by varying the temperature one can widen the time interval within which the echo effect exists.

5. Summary and discussion

In this paper we have shown that an echo effect can be found in systems consisting of self-excited oscillators. It is shown in section 2 that if the action of an external signal on oscillator phases depends on these phases, a pair of such pulses can lead to an echo response of the oscillator system. An example of such a signal is a synchronization signal for a self-excited oscillator system. In the general case the echo response has a complex form and may consist of a number of pulses. The dependence of a pulse amplitude on the number of pulses can be nonmonotonic, but the amplitude tends to zero when this number tends to infinity.

The role of mutual interaction between self-excited oscillators is very important. For example, mutual phase locking causes the echo effect to disappear because neither divergence nor convergence of oscillator phases can exist in this system.

The Josephson junction array is considered as a system capable of exhibiting an echo effect without noise. We show that an echo effect really does occur in this system under certain conditions, and that it shows up as an infinite series of pulses. In fact, the pulse amplitude tends to zero when the number of pulses tends to infinity, and only a finite number may be observable. The amplitude of the pulses is a nonmonotonic function of number, so the amplitude of an earlier echo pulse is not always larger than that of the next one: some pulses (including the first one) can even have zero amplitude. The amplitude and shape of the echo signals are obtained. The amplitudes of the echo signals depend on the amplitude and duration of the second external signal; the shapes of the echo signals depend on the frequency distribution function of a Josephson junction at a fixed bias current. This result is obtained for a large range of conditions but will be qualitatively the same even when the conditions (except for the weak-interconnection condition) mentioned above are not satisfied.

The influence of a thermal delta-correlated classical noise is estimated and the time at which the echo effect observation is possible is found. We show that all the necessary conditions can be satisfied.

It is important to note that for a self-excited oscillator system the thermal noise is the only thing that can limit a maximum time distance between external signals in the experimental observation of an echo effect. For a conservative oscillator system there is another time limit: the energy dissipation time in the system.

Acknowledgments

The authors would like to thank A A Andronov and the employees of the Superconductor Physics Department at the Institute for Physics of Microstructure of the Russian Academy of Science for fruitful discussions. This work was supported by Russian Foundation for Fundamental Research, Grant No 93-02-14718.

References

[1] Hahn E L 1950 *Phys. Rev.* **80** 580
[2] Gould R W, O'Neil T M and Malmberg J H 1967 *Phys. Rev. Lett.* **19** 219
[3] Hill R M and Kaplan D E 1965 *Phys. Rev. Lett.* **14** 1061
Gould R W 1965 *Phys. Lett.* **19** 477
[4] Abella I D, Kurnit N A and Hartman S R 1966 *Phys. Rev.* **141** 391
[5] Prigodin V N, Altshuler B L, Efetov K B and Iida S 1994 *Phys. Rev. Lett.* **72** 546
[6] Yacubovich E I 1968 *Sov. Phys.–JETP* **55** 304
[7] Landau L D 1946 *Sov. Phys.–JETP* **16** 574
[8] Hadley P, Beasley M R and Wiesenfeld K 1988 *Appl. Phys. Lett.* **52** 1619
[9] Hamilton C A, McDonald D G, Sauvageau J E and Whiteley S R 1989 *Proc. IEEE* **77** 1224
[10] Richards P L and Hu Q 1989 *Proc. IEEE* **77** 1233
[11] Han S, Bi B, Zhang W and Lukens J E 1994 *Appl. Phys. Lett.* **64** 1424
[12] Likharev K K and Kuzmin L S 1977 *Radio Eng. Electron. Phys.* **22** 109
[13] Shapiro S, Janus A R and Holly S 1964 *Rev. Mod. Phys.* **36** 223
[14] Larkin A I and Ovchinnikov Yu N 1967 *Sov. Phys.–JETP* **53** 2159